I Homotopy Theory

A. Homotopy classes of maps

Recall if X, Y are topological spaces, then two maps

$$f, g : X \to Y$$

are homotopic, denoted $f \simeq g$, if there is a continuous map

$$\Phi : X \times [0,1] \to Y$$

such that $f(x) = \Phi(x,0)$ and $g(x) = \Phi(x,1)$

note: Φ gives a one parameter family of maps

$$\phi_t : X \to Y : x \mapsto \Phi(x,t)$$

from f to g, call this a homotopy from f to g

example: X any space

$f : X \to [0,1]$ is homotopic to the constant map $g(x) = 0$, by the homotopy

$$\phi_t(x) = (1-t)\, f(x)$$

let $C(X,Y)$ = set of continuous maps from X to Y

$$[X,Y] = C(X,Y)/\simeq$$

(where homotopic maps are identified)

example: $[X, [0,1]] = \{g(x) = 0\}$

we call X a pointed space if it has a base point $x_0 \in X$

given pointed spaces $(X, x_0), (Y, y_0)$

let

$[X, Y]_0$ = homotopy classes of continuous maps $f: X \to Y$ taking x_0 to y_0

If $f: X \to X'$ is a continuous map then it induces

$$[X', Y] \xrightarrow{f^*} [X, Y]$$
$$h: X' \to Y \longmapsto h \circ f: X \to Y$$

(check this is well-defined!)

and

$$[Y, X] \xrightarrow{f_*} [Y, X']$$
$$h: Y \to X \longmapsto f \circ h: Y \to X'$$

so $[X, \cdot]: \mathcal{T}op \to \mathcal{S}et$ is a covariant functor

$[\cdot, X]: \mathcal{T}op \to \mathcal{S}et$ is a contravariant functor

$f: X \to Y$ is a homotopy inverse of $g: Y \to X$ if $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$

if g has a homotopy inverse then it is called a homotopy equivalence and X and Y are called homotopy equivalent, denote $X \simeq Y$

examples:

① $X = S^1$, $Y = S^1 \times [0,1]$ are homotopy equivalent

$f: X \to Y : \theta \mapsto (\theta, 0)$

$g: Y \to X : (\theta, t) \mapsto \theta$

$g \circ f : S^1 \to S^1$ is id_{S^1}

$f \circ g\,(\theta, s) = (\theta, 0)$ and

$$\phi_t : S^1 \times [0,1] \to S^1 \times [0,1]$$
$$(\theta, s) \longmapsto (\theta, (1-t)s)$$

is a homotopy $\phi_0 = id_Y$ to $\phi_1 = f \circ g$

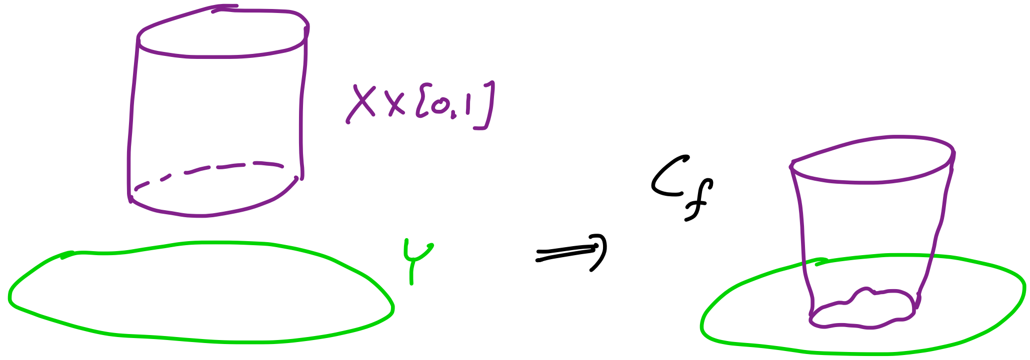
② X, Y any spaces

$f: X \to Y$ a continuous map

the mapping cylinder is

$$C_f = (X \times [0,1]) \cup Y /\sim$$

where $(x, 0) \sim f(x)$



exercise: $C_f \simeq Y$

indeed, show $\pi: C_f \longrightarrow Y$

$(x,t)\in X\times[0,1] \longmapsto f(x)$

$y \in Y \longmapsto y$

is a homotopy equivalence with homotopy inverse $i: Y \to C_f$ inclusion

there is an obvious inclusion

$$j: X \longrightarrow C_f : x \longmapsto (x,1)$$

exercise: $j \simeq i\circ f$

Slogan: $\boxed{\text{any map is an inclusion (up to homotopy)}}$

a pointed space (Y, y_0) is an H-space if there exist continuous maps

$\mu : Y \times Y \to Y$ and
$\nu : Y \to Y$

such that

① $\mu \circ \iota_1 \simeq id_Y$ and $\mu \circ \iota_2 \simeq id_Y$

where $\iota_1 : Y \to Y \times Y : y \mapsto (y, y_0)$
$\iota_2 : Y \to Y \times Y : y \mapsto (y_0, y)$

② the compositions

$$Y \times (Y \times Y) \xrightarrow{id_Y \times \mu} Y \times Y \xrightarrow{\mu} Y$$

and

$$(Y \times Y) \times Y \xrightarrow{\mu \times id_Y} Y \times Y \xrightarrow{\mu} Y$$

are homotopic

③ the composition

$$Y \xrightarrow{id_Y \times \nu} Y \times Y \xrightarrow{\mu} Y$$

and

$$Y \xrightarrow{\nu \times id_Y} Y \times Y \xrightarrow{\mu} Y$$

are homotopic to constant maps

if G is a topological group (i.e. a group with a topology such that product and inverse are continuous) then

(G, e) is an H-space

identity element

We will see more examples later but first

Th^{m} 1:

> the set $[X,Y]_0$ has a natural group structure for all pointed spaces X
>
> $\Longleftrightarrow$
>
> Y is an H-space

natural means if $f: X \to X'$ is continuous then the induced map

$$f^*: [X', Y]_0 \to [X, Y]_0$$

is a homeomorphism

Proof: ($\Leftarrow$) Suppose Y is an H-space

given any (X, x_0) notice that $\mu: Y \times Y \to Y$ gives a map

$$\mu_*: [X, Y \times Y]_0 \to [X, Y]_0$$

there is a map

$$\begin{aligned}\phi: [X,Y]_0 \times [X,Y]_0 &\longrightarrow [X, Y \times Y] \\ ([f], [g]) &\longmapsto [f \times g]\end{aligned}$$

exercise: check Φ is well-defined and a bijection

so we get "multiplication"

$$m = \mu \circ \phi : [X,Y]_0 \times [X,Y]_0 \to [X,Y]_0$$

denote $m([f],[g])$ by $[f]\cdot[g]$

similarly ν gives

$\nu_* : [X,Y]_0 \to [X,Y]_0$ and

denote $\nu_*([f])$ by $[f]^{-1}$

property ① says the constant map $c(x) = y_0$ satisfies

$$[c]\cdot[g] = [g] = [g]\cdot[c]$$

② shows $[f]\cdot([g]\cdot[h]) = ([f]\cdot[g])\cdot[h]$

③ shows $[f]^{-1}\cdot[f] = [c] = [f]\cdot[f]^{-1}$

also easy to see $f: X \to X'$ induces a homomorphism of groups ✓

($\Rightarrow$) Suppose $[X,Y]_0$ has a natural group structure $\forall X$

take $X = Y \times Y$

let $p_1, p_2 : Y \times Y \to Y$ be projection onto $1^{st}, 2^{nd}$ factor

$[p_1], [p_2] \in [Y \times Y, Y]_0$

let $[\mu] = [p_1] \cdot [p_2] \in [Y \times Y, Y]_0$

and $\mu : Y \times Y \to Y$ be any map in $[\mu]$

let $[\nu] = [id_Y]^{-1} \in [Y, Y]_0$ and $\nu : Y \to Y$

any map in $[\nu]$

we now check ①

$$i_1 : Y \to Y \times Y : y \mapsto (y, y_0)$$

induces

$$i_1^* : [Y \times Y, Y]_0 \to [Y, Y]_0$$
$$[p_1] \mapsto [p_1 \circ i_1] = [id_Y]$$
$$[p_2] \mapsto [p_2 \circ i_1] = [const]$$

naturality gives $i_1^*([p_1] \cdot [p_2]) = [id_Y] \cdot [const] = [id_Y]$

(note $[const]$ is the identity in $[Y, Y]_0$

since let $X =$ 1-pt space

$$f : Y \to X \qquad g : X \to Y : x_0 \mapsto y_0$$

$$[X, Y]_0 \xrightarrow{f_*} [Y, Y]$$
$$[g] \longmapsto [const]$$

only elt! so identity $\therefore$ identity)

so $i_1^*([\mu]) = [id_Y] \Rightarrow \mu \circ i_1 \simeq id_Y$

similarly for $\mu \circ \iota_2 \simeq id_Y$

can similarly check ② and ③ ▦

If (Y, y_0) is a pointed space, then the loop space of Y is

$$\Omega(Y) = C(([0,1], \{0,1\}), (Y, y_0))$$
$$= \{ f \in C([0,1], Y) \text{ s.t. } f(0) = f(1) = y_0 \}$$

or equivalently for $x_0 \in S^1$

$$\Omega(Y) = C((S^1, x_0), (Y, y_0))$$

lemma 2:

$$\boxed{\Omega(Y) \text{ is an } H\text{-space}}$$

Proof: define $\mu : \Omega(Y) \times \Omega(Y) \to \Omega(Y)$

by $\mu(f,g)(t) = \begin{cases} f(2t) & 0 \leq t \leq 1/2 \\ g(2t-1) & 1/2 \leq t \leq 1 \end{cases}$

and $\nu : \Omega(Y) \to \Omega(Y)$

by $\nu(f)(t) = f(1-t)$

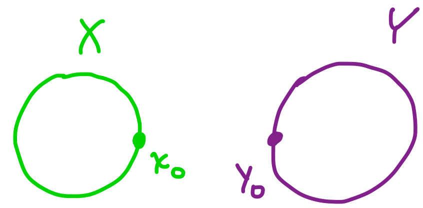
exercise: Check ①, ②, and ③ ▦

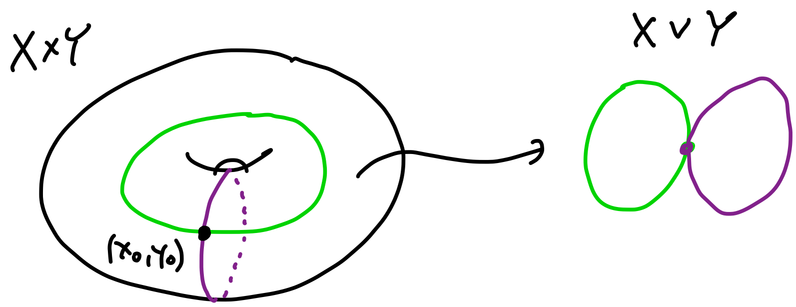
given $(X, x_0), (Y, y_0)$ their <u>wedge product</u> is

$$X \vee Y = (X \times \{y_0\}) \cup (\{x_0\} \times Y) \subset X \times Y$$

with base point (x_0, y_0)

example:





a pointed space (Y, y_0) is an <u>H'-space</u> if there are

maps $\quad \mu : Y \to Y \vee Y$ and

$\qquad\quad \nu : Y \to Y$

such that ① $p_1 \circ \mu \simeq id_Y$ and $p_2 \circ \mu \simeq id_Y$

where $p_1, p_2 : Y \vee Y \to Y$ are

projections onto the 1^{st} and 2^{nd} factors

② the compositions

$$Y \xrightarrow{\mu} Y \vee Y \xrightarrow{id_Y \vee \mu} Y \vee (Y \vee Y)$$

and
$$Y \xrightarrow{\mu} Y \vee Y \xrightarrow{\mu \vee id_Y} (Y \vee Y) \vee Y$$

are homotopic

③ the compositions

$$Y \xrightarrow{\mu} Y \vee Y \xrightarrow{a} Y$$

$$Y \xrightarrow{\mu} Y \vee Y \xrightarrow{b} Y$$

are homotopic to the constant map

where a on 1^{st} Y is ν and is the identity on 2^{nd}
b on " " id_Y " " ν on 2^{nd}

Th^m 3:

the set $[Y, X]_0$ has a natural group structure for all for all pointed spaces (X, x_0)

$\Longleftrightarrow$

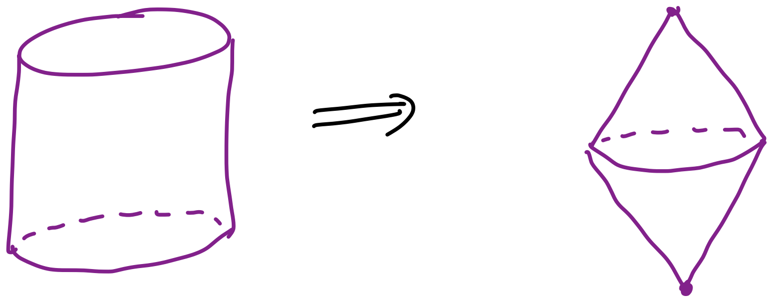
Y is an H'-space

Proof: exercise ▨

given a space X its suspension is

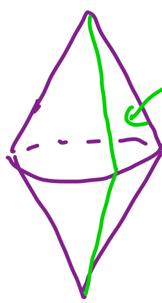
$$\Sigma X = X \times [0,1] / \sim$$

where $X \times \{0\}$ and $X \times \{1\}$ are collapsed to distinct points



if (X, x_0) is a pointed space then its suspension

is $$\Sigma X = X\times[0,1] \Big/ \{X\times\{0\},\ X\times\{1\},\ \{x_0\}\times[0,1]\}$$



collapse

this is naturally a pointed space

examples:

① $S^n = \Sigma S^{n-1} = S^{n-1}\times[0,1] \Big/ \{S^{n-1}\times\{0\},\ S^{n-1}\times\{1\}\}$

② a little harder

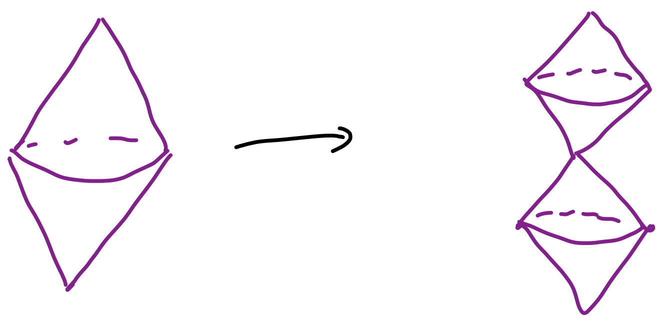
$$(S^n, x_0) = \Sigma(S^{n-1}, x_0)$$

exercise: If M a manifold and C an embedded arc prove $M/_C$ is homeomorphic to M

lemma 4:

$\boxed{\text{for any pointed space } (Y, y_0) \text{, its suspension } \Sigma Y \text{ is an } H'\text{-space}}$

Proof: define $\mu : \Sigma Y \to \Sigma Y \vee \Sigma Y$ by

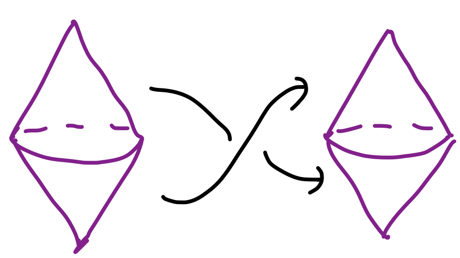


note: $\Sigma Y \vee \Sigma Y = Y \times [0,1] \big/ \{Y \times \{0\}, Y \times \{1\}, Y \times \{1/2\}, \{y_0\} \times [0,1]\}$

so μ is just collapse $Y \times \{1/2\}$

and set $\nu : \Sigma Y \to \Sigma Y$

$$(y,t) \mapsto (y, 1-t)$$



exercise: check 1, 2, and 3 ▦

Thm 5:

> If X is an H'-space and Y is an H-space then the product structures on $[X,Y]_0$ agree and are commutative

Proof:

denote the product from H'-space by $+$
and from H-space by $\bullet$

let f_1, f_2 be maps representing elts of $[X,Y]_0$

consider

$$\begin{tikzcd} & X \vee X \arrow[r, "f_1 \vee f_2"] \arrow[d, "i"] & Y \vee Y \arrow[dr, "\nabla"] & \\ X \arrow[ur, "\mu_\vee"] \arrow[dr, "\Delta"'] & & & Y \\ & X \times X \arrow[r, "f_1 \times f_2"] & Y \times Y \arrow[ur, "\mu_\times"'] \arrow[from=uu, "j"] & \end{tikzcd}$$

μ on X ; μ on Y

$\Delta(x) = (x,x)$ is diagonal map

$\nabla(y, y_0) = y = \nabla(y_0, y)$

we know $[f_1] \cdot [f_2] = [\mu_\times \circ (f_1 \times f_2) \circ \Delta]$ and

$[f_1] + [f_2] = [\nabla \circ (f_1 \vee f_2) \circ \mu_\vee]$

condition ① of H'-space says $i \circ \mu_\vee \simeq \Delta$

① of H-space says $\mu_\times \circ j \simeq \nabla$

also note

$(x, x_0) \in X \vee X$ then $i(x, x_0) = (x, x_0) \in X \times X$

and $(f_1 \times f_2) \circ i(x, x_0) = (f_1(x), y_0)$

and $f_1 \vee f_2 (x, x_0) = (f_1(x), y_0)$

$j \circ (f_1 \vee f_2) = (f_1(x), y_0)$

and similarly for (x_0, x) so
center square commutes

$$\text{so } \nabla \circ (f_1 \vee f_2) \circ \mu_V \simeq \mu_X \circ j \circ (f_1 \vee f_2) \circ \mu_V$$

$$= \mu_X \circ (f_1 \times f_2) \circ i \circ \mu_V$$

$$\simeq \mu_X \circ (f_1 \times f_2) \circ \Delta$$

$\therefore$ product structures same!

to see the product structure is commutative
we use

Fact: if $\rho : G \times G \to G : (g, h) \mapsto gh$ is
a homomorphism, then G is commutative

indeed:

$$\rho((g,h) \cdot (g^{-1}, h^{-1})) = \rho(gg^{-1}, hh^{-1}) = \rho(e,e) = e$$

$$\rho(g,h)\, \rho(g^{-1}, h^{-1}) = ghg^{-1}h^{-1}$$

so $gh = hg$ ✓

now $\mu_X : Y \times Y \to Y$ induces a homomorphism

$$[X, Y\times Y]_0 \xrightarrow{\mu_*} [X,Y]_0$$

we also have the bijection

$$\begin{aligned} \phi : [X,Y]_0 \times [X,Y]_0 &\longrightarrow [X, Y\times Y]_0 \\ ([f],[g]) &\longmapsto [f\times g] \end{aligned}$$

is a homomorphism since X is an H'-space

to see this let $p_i : Y\times Y \to Y$ be projection to the i^{th} factor

this induces a homomorphism

$$(p_i)_* : [X, Y\times Y]_0 \to [X,Y]_0$$

and ϕ is clearly the inverse of $(p_1)_* \times (p_2)_*$ so $(p_1)_* \times (p_2)_*$ is an isomorphism with inverse ϕ

$$\begin{aligned} \therefore\ \mu_* \circ \phi : [X,Y]_0 \times [X,Y]_0 &\longrightarrow [X,Y]_0 \\ ([f],[g]) &\longmapsto [f]\cdot[g] \end{aligned}$$

is a homomorphism and hence $[X,Y]_0$ is abelian from above fact ▦

lemma 6:

If Y is locally compact and Hausdorff there is a bijection $C^\circ(X\times Y, Z) \longrightarrow C^\circ(X, C^\circ(Y,Z))$

$$g: X\times Y \to Z \longmapsto \psi: X \to C^\circ(Y,Z)$$
$$x \mapsto g(x,\cdot): Y \to Z$$

$$\phi: X\times Y \to Z \longleftarrow f: X \to C^\circ(Y,Z)$$
$$(x,y) \mapsto f(x)(y)$$

if X is also Hausdorff this is a homeomorphism

a space is locally compact if every point has a nbhd U that is contained in a compact set.

Remark: lemma implies $[X\times Y, Z] = [X, C^\circ(Y,Z)]$

Proof: to prove this lemma we need a topology on $C^\circ(X,Y)$

we use the compact open topology

given C a compact set in X and W an open set in Y

set $U(C,W) = \{f \in C^\circ(X,Y) : f(C) \subset W\}$

this forms a subbasis for a topology on $C^\circ(X,Y)$ called the compact open topology exercisé:

1) if Y is a metric space, show this is the "topology of compact convergence"

i.e. $f_n \to f$ iff $\forall$ compact $C \subset X$, $f_n|_C \to f|_C$ uniformly

2) If $f: X \times Y \to Z$ is continuous, then so is $F: X \to C^0(Y, Z) : x \mapsto f_x : Y \to Z$, $y \mapsto f(x,y)$ also continuous

3) the converse is true if Y is locally compact and Hausdorff

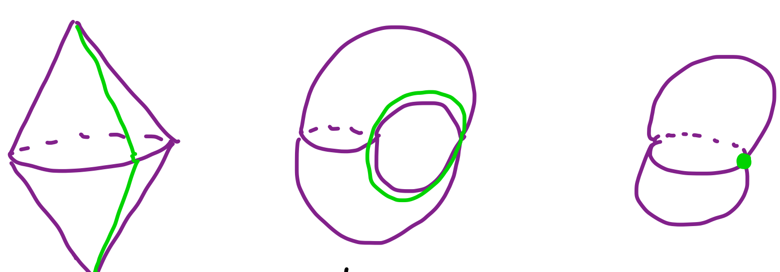
4) prove the theorem

the <u>smash product</u> of two pointed spaces X, Y is

$$X \wedge Y = X \times Y / X \vee Y = X \times Y / X \times \{y_0\} \cup \{x_0\} \times Y$$

note the reduced suspension is

$$\Sigma X = S^1 \wedge X = S^1 \times X / \{pt\} \times X \cup S^1 \times \{x_0\}$$



$S^1 \times X / \{pt\} \times X$

Corollary 7:

If Y is locally compact and Hausdorff
$$[X \wedge Y, Z]_0 = [X, C^0_{based}(Y, Z)]_0$$

Proof: if $f \in C^0_{based}(X, C^0_{based}(Y, Z))$

then $f(x_0) =$ constant map $Y \to Z : y \longmapsto z_0$

so the map $F : X \times Y \to Z : (x, y) \to f(x)(y)$

sends $\{x_0\} \times Y \longmapsto z_0$

also $f(x) : Y \to Z$ that sends y_0 to z_0

so $F(X \times \{y_0\}) = \{z_0\}$

and F induces a map in $[X \wedge Y, Z]_0$

you can similarly construct the inverse map $\blacksquare$

recall: the loop space is $\Omega(X) = C^0_{based}(S^1, X)$

Corollary 8:

$$[\Sigma X, Y]_0 = [X, \Omega(Y)]_0$$

"suspension is the adjoint of looping"

Proof:

$$[\Sigma X, Y]_0 = [S^1 \wedge X, Y]_0 = [X, C^0_{based}(S^1, Y)]_0$$
$$= [X, \Omega(Y)]_0$$

the n^{th} homotopy group of a based space (X, x_0) is

$$\pi_n(X) = [S^n, X]_0$$

note: 1) $\pi_n(X) = [S^n, X]_0 = [\Sigma S^{n-1}, X]_0$
$$= [S^{n-1}, \Omega(X)]_0$$
$$= \dots = [S^1, \Omega^{n-1}(X)] = \pi_1(\Omega^{n-1}(X))$$

so $\boxed{\pi_n(X) = \pi_1(\Omega^{n-1}(X))}$

and $\boxed{\pi_n(X) = \pi_0(\Omega^n(X))}$

recall $\pi_0(Y) = \#$ path components

so $\pi_n(X) = \#$ of path components of $\Omega^n(X)$

2) $\pi_n(X) = [S^{n-1}, \Omega(X)]_0$

$n \geq 2$, $\Omega(X)$ an H-space an S^{n-1} an H'-space so Th^{m} 5 gives

$\boxed{\pi_n(X) \text{ is abelian for } n \geq 2}$

3) if X is a Lie group, then it is an H-space

so $\pi_1(X)$ is abelian

How does $[X,Y]_0$ depend on the base points?

given $f_0, f_1 : X \to Y$ and

$u : [0,1] \to Y$

if there is a homotopy $F : X \times [0,1] \to Y$ s.t.

$$F(x,0) = f_0(x)$$
$$F(x,1) = f_1(x)$$
$$F(x_0,t) = u(t)$$

then we say f_0 and f_1 are homotopic along u

and denote this $f_0 \simeq_u f_1$

if f_0 and f_1 preserve base points then

u is a loop in Y

to have nice properties when we move the base point, we need (X, x_0) to be "non-degenerate" by which we mean they are an NDR-pair

Neighborhood Deformation Retract

$A \subset X$ is an NDR-pair if there are maps

$u : X \to [0,1]$

$h : X \times [0,1] \to X$

such that

1) $A = u^{-1}(0)$

2) $h(\cdot, 0) = id_X(\cdot)$

3) $h(a,t) = a \quad \forall\, t \in [0,1],\ a \in A$

4) $h(x,1) \in A \quad \forall\, x \in X$ with $u(x) < 1$

(so the nbhd $u^{-1}([0,1))$ of A deformation retracts to A)

exercise:

1) if A is a sub-CW complex of X then (X,A) is an NDR-pair

2) if A is a submanifold of X then (X,A) is an NDR-pair

in particular $A = pt$ in S^n is an NDR-pair

lemma 9:

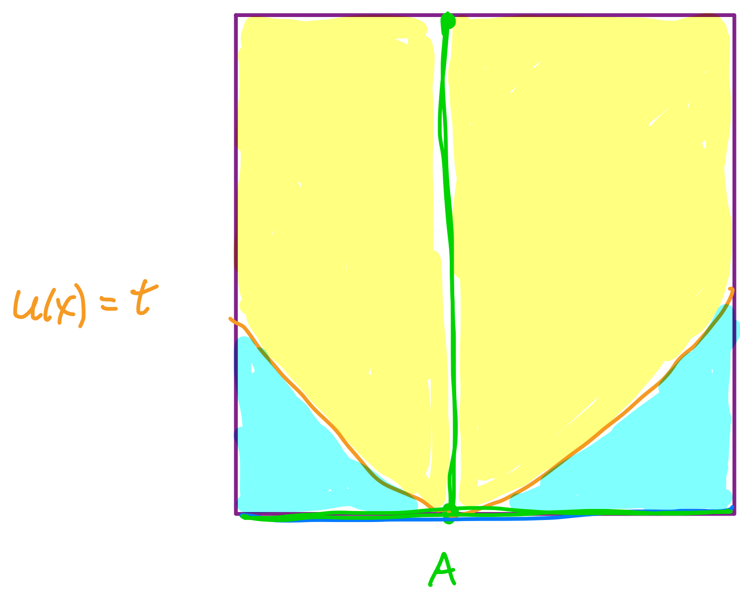
If (X,A) is an NDR-pair, then $(X \times \{0\}) \cup (A \times [0,1])$ is a retract of $X \times [0,1]$

Proof: set

$$R : X \times [0,1] \longrightarrow (X \times \{0\}) \cup (A \times [0,1])$$

to be

$$R(x,t) = \begin{cases} (x,t) & x \in A \text{ or } t = 0 \\ (h(x,1), t-u(x)) & t \geq u(x) \text{ and } t > 0 \\ \left(h\left(x, \frac{t}{u(x)}\right), 0\right) & u(x) \geq t \text{ and } u(x) > 0 \end{cases}$$



exercise: check this is a retract

lemma 10:

$$\boxed{\begin{array}{l} \text{If } (X, x_0) \text{ an NDR-pair and} \\ \quad f_0 : X \to Y \quad \text{with } f_0(x_0) = y_0 \\ \quad \gamma : [0,1] \to Y \text{ a path } y_0 \text{ to } y_1 \\ \text{then } \exists \text{ an } f_1 : X \to Y \text{ such that} \\ \qquad f_1(x_0) = y_1 \text{ and } f_0 \simeq_\gamma f_1 \end{array}}$$

denote f_1 by $\gamma \cdot f_0$ (well-defined once R fixed)

Proof: let R be as defined in the last proof for (X, x_0) so R is a retract

$$R: X \times [0,1] \longrightarrow (X \times \{0\}) \cup (\{x_0\} \times [0,1])$$

now let $H: X \times [0,1] \longrightarrow Y$ be

$$H(x,t) = \begin{cases} f_0(R(x,t)) & \text{if } R(x,t) \in X \times \{0\} \\ \gamma(R(x,t)) & \text{if } R(x,t) \in \{x_0\} \times [0,1] \end{cases}$$

let $f_1(x) = H(x,1)$

Clearly H is a homotopy along γ between f_0 and f_1 ▦

lemma 11:

Suppose $f_0, f_1, f_2 : X \to Y$ are based maps, and (X, x_0) is an NDR-pair.

If $f_0 \simeq_\gamma f_1$ and $f_0 \simeq_{\gamma'} f_2$ with $\gamma \simeq \gamma'$ rel boundary then $f_1 \simeq f_2$ rel base point

Idea of Proof: (X, x_0) an NDR-pair then you can show $\left(X \times [0,1], (X \times \{0,1\}) \cup (\{x_0\} \times [0,1]) \right)$ is an NDR-pair so there is a retraction

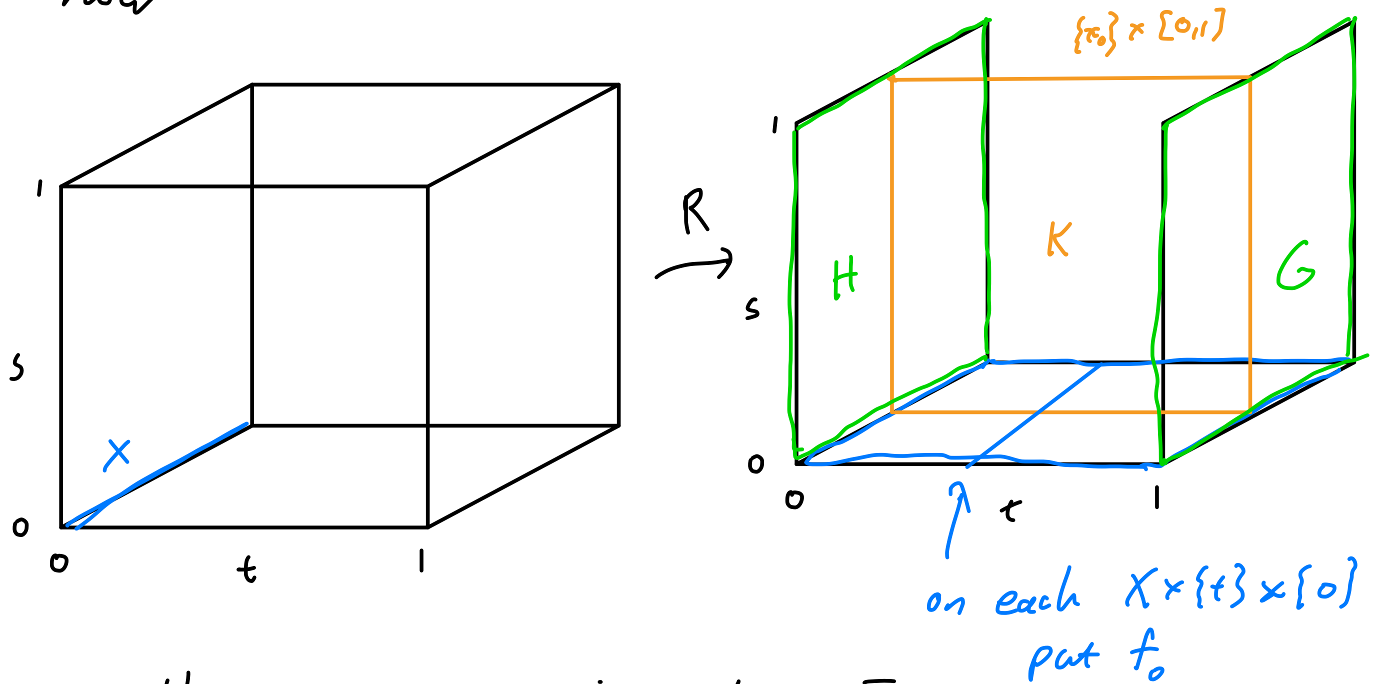
$$R: X \times [0,1] \times [0,1] \longrightarrow (X \times [0,1]) \times \{0\} \cup ((X \times \{0,1\}) \cup (\{x_0\} \times [0,1])) \times [0,1]$$

let H be the homotopy $f_0 \sim_\gamma f_1$,

G " " $f_0 \sim_{\gamma'} f_2$, and

K " " $\gamma \sim \gamma'$ rel end pts

now



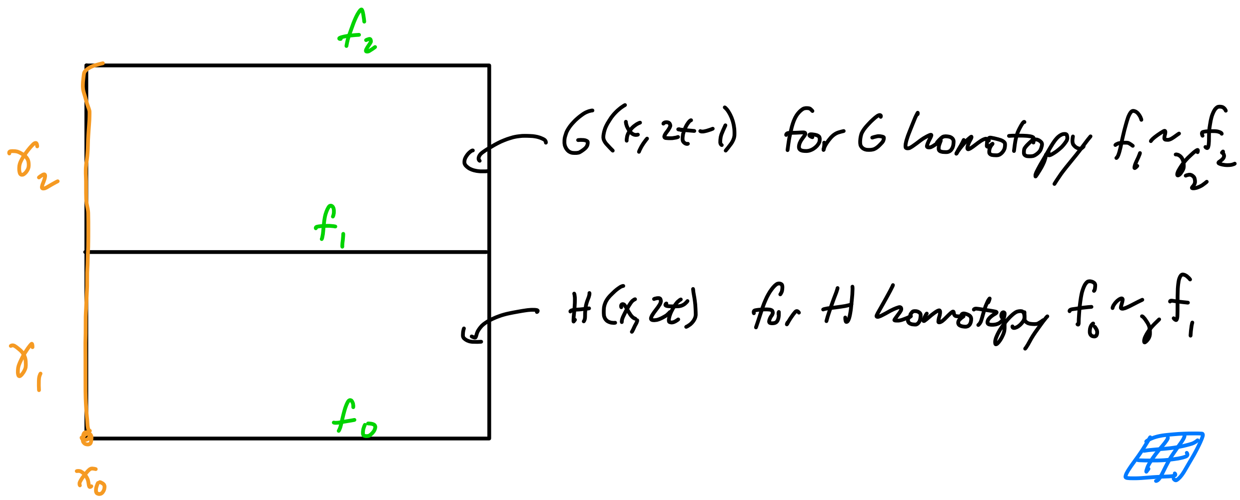
call the composition above $\bar{R}$

$\bar{R}\big|_{X \times [0,1] \times \{1\}}$ is the homotopy $f_1 \sim f_2$ rel base pt

lemma 12:

> Suppose $f_0 : X \to Y$ and $f_1 : X \to Y$ are homotopic along γ_1, $f_2 : X \to Y$ is homotopic to f_1 along γ_2 and $\gamma_1(1) = \gamma_2(0)$, then $f_0 \sim_{\gamma_1 * \gamma_2} f_2$

Proof: just concatenate the homotopies



Thm 13:

> If x_0 is a nondegenerate base point of X then
> $\pi_1(Y, y_0)$ acts on $[X, Y]_0$
> moreover, $[X, Y]$ is the quotient set of $[X, Y]_0$ by the $\pi_1(X, x_0)$ action if Y is path connected

Proof: $[\gamma] \in \pi_1(Y, y_0)$ and $[f] \in [X, Y]_0$

by lemma 10 we have $\gamma \cdot f : X \to Y$
and $[\gamma \cdot f]$ clearly in $[X, Y]_0$

Claim: $[\gamma \cdot f]$ is well-defined

suppose $f, g \in [f]$ so $f \simeq g$ rel base pt
by lemma 10 $\exists$ f_1, g_1 such that
$f \simeq_\gamma f_1$ and $g \simeq_\gamma g_1$

thus $f_1 \simeq_{\gamma^{-1}} f \simeq g \simeq_\gamma g_1$
rel base point

by lemma 12 $f_1 \simeq_{\gamma^{-1} * \text{const} * \gamma} g_1$

and lemma 11 says $f_1 \simeq g_1$ rel base boint

so $[\gamma \cdot f] = [\gamma \cdot g]$ and $[\gamma \cdot f]$ does not depend on $f \in [f]$

and $[\gamma \cdot f]$ does not depend on $\gamma \in [\gamma]$ by lemma 11

let $\Phi : [X, Y]_0 \to [X, Y]$ be the map that forgets base pt.

clearly $\Phi([\gamma] \cdot [f]) = \Phi([\gamma])$

so Φ is invariant under $\pi_1(Y, y_0)$ action

$\therefore$ Φ defines a map $[X, Y]_0 / \pi_1(Y, y_0) \to [X, Y]$

if $\Phi([f]) = \Phi([g])$, then

let H be the free homotopy f to g

and $\gamma(t) = H(x_0, t)$

f, g both take x_0 to y_0 so $\gamma \in \pi_1(Y, y_0)$

and $[\gamma] \cdot [f] = [g]$

so Φ is injective on $[X, Y]_0 / \pi_1(Y, y_0)$

Φ is clearly onto by lemma 10 if Y is path connected

Corollary 14:

$\boxed{\begin{array}{c}\text{a based map is null-homotopic} \\ \Longleftrightarrow \\ \text{it is based null-homotopic}\end{array}}$

Proof: clearly based null-homotopic $\Rightarrow$ null-homotopic

now if $f \simeq c$ (the constant map, can assume $c(x) = y_0$) by homotopy H

let $\gamma(t) = H(x_0, t)$ and we see $f \simeq_\gamma c$

so $c \simeq_{\gamma^{-1}} f$

of course $c \simeq_{\gamma^{-1}} c$

so $f \simeq c$ rel base point by lemma 11 $\blacksquare$

Corollary 15:

$\boxed{\begin{array}{c}\text{if } Y \text{ is simply connected then} \\ [X, Y]_0 \to [X, Y] \\ \text{is a bijection}\end{array}}$

note: a special case of Th^m 13 says

$\pi_1(Y, y_0)$ acts on $\pi_n(Y, y_0)$

with quotient $[S^n, Y]$

(if Y is connected)